Supplementary online material for "Solving Set Cover with Pairs Problem using quantum annealing"

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Details of the example SCP instance

In the paper we consider an example SCP instance for illustrating our mappings from SCP to Ising and eventually to a Chimera graph. Specifically, the $\textsc{Ising}(\mathbf{h}, \mathbf{J})$ described in Figure 2b has

$$\mathbf{h}^T = \frac{1}{8}\cdot \begin{array}{c} \begin{array}{cccccccccccccc} s_1 & s_2 & s_3 & s_4 & t_{12}^{(1)} & t_{14}^{(1)} & t_{24}^{(1)} & t_{13}^{(2)} & t_{14}^{(2)} & t_{34}^{(2)} & x_1^{(1)} & x_2^{(1)} & x_1^{(2)} & x_2^{(2)} \end{array} \\ \left(\begin{array}{cccccccccccccc} 7 & 3 & 3 & 7 & -6 & -6 & -6 & -6 & -6 & -6 & 2 & 4 & 2 & 4 \end{array}\right) \end{array}.$$

Here the labels above each element of $\mathbf{h}$ indicates which spin the coefficient is associated to. The matrix of interaction coefficients $\mathbf{J}$ is shown in Figure S1.

Proof of correctness for Algorithm 1

Here we show that Algorithm 1 indeed samples uniformly from all $(2^n-1)^m$ possible "dummy-free" bipartite graphs for a fixed setting of the ground set U of size n and cover set S of size m. Formally we say a bipartite graph $G(U\cup S, E)$ between two sets U and S is *dummy-free* if for any $s\in S$ there exists at least one $u\in U$ such that $(s,u)\in E$. Then we state the following claim.

Claim. *Given any set U of n elements and S of m elements, for any dummy-free bipartite graph $G(V,E)$ between U and S, Algorithm 1 generates G with probability $(2^n-1)^{-m}$.*

Proof. Let $\Pr(G)$ be the probability that Algorithm 1 generates G. Recall that if at a particular $s\in S$ during the looping on line 2, when Algorithm 1 scanned through all $u\in U$ but did not end up selecting any element in U, the algorithm enters line 7 to repeat the process from scratch for s. Then depending on how many times the algorithm entered line 7 during the process of generating G, we could express $\Pr(G)$ as

$$\Pr(G) = \sum_{k=0}^{\infty} \Pr(G|\text{Algorithm 1 entered line 7 in total } k \text{ times}) \tag{S2}$$

If the algorithm never entered line 7 and generated G, then the probability of generating G is essentially the probability of mn coin flips, namely 2^{-mn}. If the algorithm entered line 7 once, then the probability $\Pr(G) = 2^{-mn}\cdot m2^{-n}$, where the extra factor $m2^{-n}$ is essentially the probability of one hit and $m-1$ misses during m

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$$
\mathbf{J} \;=\; \frac{1}{8}\cdot
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} s_1 & s_2 & s_3 & s_4 & t_{12}^{(1)} & t_{14}^{(1)} & t_{24}^{(1)} & t_{13}^{(2)} & t_{14}^{(2)} & t_{34}^{(2)} & x_1^{(1)} & x_2^{(1)} & x_1^{(2)} & x_2^{(2)} \end{array} \\
\begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \\ t_{12}^{(1)} \\ t_{14}^{(1)} \\ t_{24}^{(1)} \\ t_{13}^{(2)} \\ t_{14}^{(2)} \\ t_{34}^{(2)} \\ x_1^{(1)} \\ x_2^{(1)} \\ x_1^{(2)} \\ x_2^{(2)} \end{array} &
\begin{pmatrix}
 & & & & -1 & -1 & & -1 & -1 & & & & & \\
 & & & & -1 & & -1 & & & & & & & \\
 & & & & & & & -1 & & -1 & & & & \\
 & & & & & -1 & -1 & & -1 & -1 & & & & \\
-1 & -1 & & & & 1 & & & & & -2 & & & \\
-1 & & & -1 & 1 & & & & & & -2 & & & \\
 & -1 & & -1 & & & & & & & 1 & -2 & & \\
-1 & & -1 & & & & & & 1 & & & & -2 & \\
-1 & & & -1 & & & & 1 & & & & & -2 & \\
 & & -1 & -1 & & & & & & & & & 1 & -2 \\
 & & & & -2 & -2 & 1 & & & & & -2 & & \\
 & & & & & & -2 & & & & -2 & & & \\
 & & & & & & & -2 & -2 & 1 & & & & -2 \\
 & & & & & & & & & -2 & & & -2 &
\end{pmatrix}
\end{array}
\tag{S1}
$$

Figure S1: The matrix of coupling coefficients in the Ising Hamiltonian instance constructed for the example SCP instance shown in Figure 2. The interpretation of the matrix elements of **J** follows Definition 2.

independent Bernoulli trial with the hit probability 2^{-n} (if we regard the event of entering line 7 as a hit). Carrying this argument to the general case if the algorithm enters line 7 k times, then we need to consider all possible ways of distributing the k hits onto the m iterations on line 2. This gives

$$
\Pr(G|\text{Algorithm 1 entered line 7 in total } k \text{ times}) = \sum_{(k_1,\cdots,k_m)} 2^{-mn}\cdot \binom{k}{k_1, k_2, \cdots, k_m}\cdot (2^{-n})^{k_1+k_2+\cdots+k_m} \tag{S3}
$$

where the summation is over the set of non-negative integers k_1 through k_m that sums up to k. Then Equation S2 leads to

$$
\begin{aligned}
\Pr(G) &= 2^{-mn}\cdot \sum_{k=0}^{\infty} \sum_{(k_1,\cdots,k_m)} 2^{-kn}\binom{k}{k_1,\cdots,k_m} \\
&= 2^{-mn}(1+2^{-n}+2^{-2n}+\cdots)^m \\
&= 2^{-mn}\left(\frac{1}{1-2^{-n}}\right)^m \\
&= (2^n-1)^{-m}.
\end{aligned}
\tag{S4}
$$

□